Energy-momentum Injection Surface in the Causal Diffusion Formulation for the Source Term

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I. CAUSAL DIFFUSION FORMULATION

We assume that the diffusion process for the energy and momentum of a parton to be absorbed in the medium is modeled by the causal diffusion equation:

$$s^\mu(t,\vec{x}) = p^\mu_{\rm parton}\rho(t,\vec{x}), \tag{1}$$

$$\tau_{\rm diff}\frac{\partial^2\rho(t,\vec{x})}{\partial t^2}+\frac{\partial\rho(t,\vec{x})}{\partial t} = D_{\rm diff}\nabla^2\rho(t,\vec{x}),\ \ (t>t_{\rm start}), \tag{2}$$

where s^μ is the distribution of the diffusing momentum, $p^\mu_{\rm parton}$ is the momentum of the parton being diffused. In the Causal Liquefier module of JETSCAPE, the initial conditions are $\rho(t=t_{\rm start},\vec{x}) = \delta^{(3)}(\vec{x}-\vec{x}_{\rm parton})$ and $\frac{\partial\rho}{\partial t}(t=t_{\rm start},\vec{x})=0$. Here $t_{\rm start}$ is the Cartesian time when the energy of the parton start the evolution by the diffusion equation and $\vec{x}_{\rm parton}$ is the position of the parton. $(t_{\rm start},\vec{x}_{\rm parton})$ is denoted as (`x_drop[0]`, `x_drop[1]`, `x_drop[2]`, `x_drop[3]`) in the code.

TABLE I: Main Parameters in the Cousal Liquefier

Notation in this document	Notation in the code	Description
$\tau_{\rm diff}$	`time_relax`	Relaxation time in the diffusion equation
$D_{\rm diff}$	`d_diff`	Diffusion coefficient in the diffusion equation
$\tau_{\rm delay}$	`tau_delay`	Proper time duration of evolution by the Diffusion Equation

II. GAUSS'S LAW AND ENERGY MOMENTUM TENSOR

The causal diffusion equation can be written in a form of the equation of continuity:

$$\frac{\partial\rho(t,\vec{x})}{\partial t}+\vec{\nabla}\cdot\vec{j}(t,\vec{x})=0, \tag{3}$$

with the current $\vec{j}$ satisfying the constitutive equation,

$$\tau_{\rm diff}\frac{\partial}{\partial t}\vec{j}+\vec{j}=-D_{\rm diff}\vec{\nabla}\rho. \tag{4}$$

Denote $j^\mu=(\rho,\vec{j})$ and Eq. (3) can be re-written as $\partial_\mu j^\mu=0$. Using the Gauss's law, the conservation law can be written as

$$0 = \int dV_4\partial_\mu j^\mu = \int d\Sigma^{\rm dep.}_\mu j^\mu - \int d\Sigma^{\rm start}_\mu j^\mu, \tag{5}$$

$$\int d\Sigma^{\rm dep.}_\mu j^\mu = \int d\Sigma^{\rm start}_\mu j^\mu = \int dV_3\rho(t=t_{\rm start},\vec{x})=1, \tag{6}$$

$$d\Sigma^{\rm start}_\mu \equiv (dV_3,0,0,0). \tag{7}$$

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Here $d\Sigma_\mu^{\rm start}$ is the normal vector for the 3-dimensional hypersurface at $t = t_{\rm start}$ and $d\Sigma_\mu^{\rm dep.}$ is the normal vector for the 3-dimensional hypersurface where the energy-momentum injection into the medium happens. dV_4 is the 4-dimensional volume element covered with those hypersurfaces. Thus, an element of the deposited energy and momentum at the energy-momentum injection surface can be written as

$$dp^\mu_{\rm dep} = p^\mu_{\rm parton}(d\Sigma^{\rm dep.}_\nu j^\nu), \tag{8}$$

and the conservation law can be confirmed as

$$p^\mu_{\rm total\,dep.} \equiv \int dp^\mu_{\rm dep} = \int p^\mu_{\rm parton}(d\Sigma^{\rm dep.}_\nu j^\nu) = p^\mu_{\rm parton}, \tag{9}$$

where $p^\mu_{\rm total\,dep.}$ is the total momentum covered with the injection hypersurface. Thus, for the calculations with an arbitrary injection hypersurface $d\Sigma^{\rm dep.}_\nu$, one needs to obtain all the components of $j^\nu = (\rho, \vec{j})$. Finally, the energy momentum tensor at the injection surface $T^{\mu\nu}_{\rm dep}$ can be expressed as

$$T^{\mu\nu}_{\rm dep} d\Sigma^{\rm dep.}_\nu = \frac{p^\mu_{\rm parton}[p^0_{\rm parton} j^\nu]}{p^0_{\rm parton}} d\Sigma^{\rm dep.}_\nu = p^\mu_{\rm parton} j^\nu d\Sigma^{\rm dep.}_\nu \tag{10}$$

It should be noted that the $T^{\mu\nu}_{\rm dep}$ does not have the symmetric structure realized for the local equilibrium case.

III. ENERGY-MOMENTUM DEPOSITION AT A CONSTANT-τ HYPERSURFACE

In the MUSIC hydro module incorporated with Causal Liquefier of JETSCAPE, the energy-momentum deposition into the fluid is carried out at the constant-τ hypersurface at $\tau = \tau_{\rm dep} = \tau_{\rm start} + \tau_{\rm delay}$ [Here $\tau_{\rm start}$ is the starting proper time for the evolution by the diffusion equation and calculated by using $t_{\rm start}$ and $\vec{x}_{\rm parton}$. It is denoted by `tau_drop` in the code.] :

$$d\Sigma^{\rm dep.}_{\bar\alpha} = \Lambda_{\bar\alpha}{}^{\mu}(\eta_{\rm s})\; d\Sigma^{\rm dep.}_\mu = (\,\tau_{\rm dep}\, dx\, dy\, d\eta_{\rm s}\,, 0, 0, 0) \tag{11}$$

Here, $\bar\alpha(, \bar\beta, \bar\gamma, \ldots) = \tau, x, y, \eta_{\rm s}$ is the suffixes for the components in the τ-$\eta_{\rm s}$ coordinate system and $\Lambda^\mu_{\bar\alpha}(\eta_{\rm s})$ is the matrix for the transformation from the Cartesian coordinate system $[x^\mu = (x^0, x^1, x^2, x^3) = (t, x, y, z)]$ to τ-$\eta_{\rm s}$ coordinate system as

$$\Lambda_{\bar\alpha}{}^{\mu}(\eta_{\rm s}) = \begin{pmatrix} \cosh\eta_{\rm s} & 0 & 0 & \sinh\eta_{\rm s} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh\eta_{\rm s} & 0 & 0 & \cosh\eta_{\rm s} \end{pmatrix}. \tag{12}$$

Since

$$T^{\mu\nu}_{\rm dep} d\Sigma^{\rm dep.}_\nu = p^\mu_{\rm parton} j^\nu d\Sigma^{\rm dep.}_\nu = p^\mu_{\rm parton} j^{\bar\alpha} d\Sigma^{\rm dep.}_{\bar\alpha} = p^\mu_{\rm parton} j^\tau\, [\tau dx dy d\eta_{\rm s}]\,, \tag{13}$$

The output from the Causal Liquefier is

$$p^\mu_{\rm parton} j^\tau = p^\mu_{\rm parton} \Lambda^\tau{}_\mu(\eta_{\rm s}) j^\mu = p^\mu_{\rm parton}\,(\rho\cosh\eta_{\rm s} - j^z \sinh\eta_{\rm s})\,, \tag{14}$$

where

$$\Lambda^{\bar\alpha}{}_\mu(\eta_{\rm s}) = \begin{pmatrix} \cosh\eta_{\rm s} & 0 & 0 & -\sinh\eta_{\rm s} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\sinh\eta_{\rm s} & 0 & 0 & \cosh\eta_{\rm s} \end{pmatrix}. \tag{15}$$

The transformation by this matrix is done by the functions `get_tau` and `get_peta` in the code. Thus, in the Causal Liquefier module, ρ and j^x (`jt` and `jz` in the code) are caluculated from the relativistic diffusion equation and then gives the source term

$$\begin{aligned} J^{\bar\alpha} &= \Lambda^{\bar\alpha}{}_\mu(\eta_{\rm s}) J^\mu = \Lambda^{\bar\alpha}{}_\mu(\eta_{\rm s}) T^{\mu\nu}_{\rm dep}(t, \vec{x}) d\Sigma^{\rm dep.}_\nu \\ &= p^\mu_{\rm parton} j^\tau = p^\mu_{\rm parton}\,(\rho\cosh\eta_{\rm s} - j^z\sinh\eta_{\rm s}) \end{aligned} \tag{16}$$

[$j^\tau = (\rho \cosh\eta_s - j^z \sinh\eta_s)$ is `jtau` in the code] to the MUSIC hydro module solving the hydrodynamic equation with the source term in the τ-η_s coordinates:

$$\nabla_{\bar{\alpha}} T^{\bar{\alpha}\bar{\beta}}_{\rm fluid} \;=\; J^{\bar{\beta}}, \tag{17}$$

where

$$\nabla_{\bar{\alpha}} T^{\bar{\alpha}\bar{\beta}}_{\rm fluid} = \partial_{\bar{\alpha}} T^{\bar{\alpha}\bar{\beta}}_{\rm fluid} + \Gamma^{\bar{\alpha}}_{\bar{\alpha}\bar{\gamma}} T^{\bar{\gamma}\bar{\beta}}_{\rm fluid} + \Gamma^{\bar{\beta}}_{\bar{\alpha}\bar{\gamma}} T^{\bar{\alpha}\bar{\gamma}}_{\rm fluid} \tag{18}$$

is the covariant derivative of $T^{\bar{\alpha}\bar{\beta}}_{\rm fluid}$ in the τ-η_s coordinates. The source term is denoted as

$$J^{\bar{\alpha}} = (j^\tau, j^x, j^y, j^{\eta_{\eta_s}}) = (\texttt{jmu[0], jmu[1], jmu[2], jmu[3]}) \tag{19}$$

in the code.