Causal Diffusion Equation

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Contents

I. PREPARATION: ONE-SIDED FOURIER TRANSFORMATION

One-sided Fourier transformation and its inverse transformation are

$$f(\omega,\vec{k}) = \int_0^\infty dt \int d^3x \mathrm{e}^{i\left(\omega t - \vec{k}\cdot\vec{x}\right)} f(t,\vec{x}), \tag{1}$$

$$f(t,\vec{x}) = \int_{-\infty+i\sigma}^{\infty+i\sigma} \frac{d\omega}{2\pi} \int \frac{d^3k}{(2\pi)^3} \mathrm{e}^{-i\left(\omega t - \vec{k}\cdot\vec{x}\right)} f(\omega,\vec{k}). \tag{2}$$

II. GENERAL SOLUTION IN THE FOURIER SPACE

We would like to solve the differential equation

$$\tau \frac{\partial^2 \rho}{\partial t^2} + \frac{\partial \rho}{\partial t} \ = \ D\nabla^2 \rho, \ \ (t>0), \tag{3}$$

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with a given set of the initial condition for n and $\partial\rho/\partial t$ at $t=0$. Perform the one-sided Fourier transformation (with imposing the initial conditions),

$$\int_0^\infty dt e^{i\omega t}\left[\tau\frac{\partial^2\rho}{\partial t^2}(t,\vec{k})+\frac{\partial\rho}{\partial t}(t,\vec{k})\right] = -k^2D\rho(\omega,\vec{x}), \tag{4}$$

$$\left[\tau e^{i\omega t}\frac{\partial\rho}{\partial t}(t,\vec{k})\right]_0^\infty - i\omega\tau\left(\left[e^{i\omega t}\rho(t,\vec{k})\right]_0^\infty - i\omega\int_0^\infty dt e^{i\omega t}\rho(t,\vec{k})\right)+\left[e^{i\omega t}\rho(t,\vec{k})\right]_0^\infty - i\omega\int_0^\infty dt e^{i\omega t}\rho(t,\vec{k}) = -k^2D\rho(\omega,\vec{x}), \tag{5}$$

$$-\tau\frac{\partial\rho_0}{\partial t} - i\omega\tau\left(-\rho_0 - i\omega\rho\right) - \rho_0 - i\omega\rho = -k^2D\rho, \tag{6}$$

Then we obtain

$$\rho(\omega,\vec{k}) = \frac{[1-i\omega\tau]\,\rho_0+\tau\frac{\partial\rho_0}{\partial t}}{-\omega^2\tau - i\omega + k^2D}, \tag{7}$$

where $\rho_0 \equiv \int d^3x e^{-i\vec{k}\cdot\vec{x}}\rho(t=0,\vec{x})$ and $\frac{\partial\rho_0}{\partial t} \equiv \int d^3x e^{-i\vec{k}\cdot\vec{x}}\frac{\partial\rho}{\partial t}(t=0,\vec{x})$.

III. INITIAL CONDITIONS, CURRENT, AND GREEN'S FUNCTION

A. Initial conditions

For the case with the initial conditions, $n(t=0,\vec{x})=\delta^{(3)}(\vec{x})$ and $\frac{\partial\rho}{\partial t}(t=0,\vec{x})=0$,

$$\rho(\omega,\vec{k}) = \frac{1-i\omega\tau}{-\omega^2\tau - i\omega + k^2D}, \tag{8}$$

since $\rho_0 = \int d^3x e^{-i\vec{k}\cdot\vec{x}}\delta^{(3)}(\vec{x}) = 1$.

B. Current

The current $\vec{j}$ is defined to have the equation of continuity and satisfies the constitutive equation for the relativistic diffusion equation,

$$\tau\frac{\partial}{\partial t}\vec{j}+\vec{j} = -D\vec{\nabla}\rho. \tag{9}$$

After the Fourier transformation one can obtain,

$$\vec{j}(\omega,\vec{k}) = \frac{-i\vec{k}D}{1-i\omega\tau}\rho(\omega,\vec{k}) = \frac{-i\vec{k}D}{-\omega^2\tau - i\omega + k^2D}, \tag{10}$$

C. Green's function

In the ω-$\vec{k}$ space, the Green's function for the relativistic diffusion equation can be written as

$$G(\omega,\vec{k}) = \frac{1}{-\omega^2\tau - i\omega + k^2D}. \tag{11}$$

So, n and $\vec{j}$ can be written with G as

$$\rho(\omega,\vec{k}) = (1-i\omega\tau)\,G(\omega,\vec{k}), \qquad \vec{j}(\omega,\vec{k}) = -iD\vec{k}G(\omega,\vec{k}), \tag{12}$$

$$\rho(t,\vec{x}) = \left(1+\tau\frac{\partial}{\partial t}\right)G(t,\vec{x}), \qquad \vec{j}(t,\vec{x}) = -D\vec{\nabla}G(t,\vec{x}). \tag{13}$$

IV. CALCULATION OF THE GREEN'S FUNCTION IN t-$\vec{x}$ SPACE

Perform the inverse Fourier transformation,

$$G(t,\vec{x}) = \int\frac{d\omega}{2\pi}\int\frac{d^3k}{(2\pi)^3}\mathrm{e}^{-i\left(\omega t-\vec{k}\cdot\vec{x}\right)}G(\omega,\vec{k}) = \int\frac{d\omega}{2\pi}\int\frac{d^3k}{(2\pi)^3}\frac{\mathrm{e}^{-i\left(\omega t-\vec{k}\cdot\vec{x}\right)}}{-\omega^2\tau-i\omega+k^2D} \tag{14}$$

The solutions of the characteristic equation $\omega^2+\frac{i}{\tau}\omega-\frac{D}{\tau}k^2=0$ are $\omega_\pm=\frac{1}{2\tau}\left[-i\pm\sqrt{4Dk^2\tau-1}\right]$ and one can get

$$G(t,\vec{x}) = \int\frac{d\omega}{2\pi}\int\frac{d^3k}{(2\pi)^3}\frac{\mathrm{e}^{-i\left(\omega t-\vec{k}\cdot\vec{x}\right)}}{-\tau}\frac{1}{(\omega-\omega_+)(\omega-\omega_-)}. \tag{15}$$

Remind that the relativistic diffusion equation describe the evolution for the time $t>0$, carry out the integration in the lower half plane of ω,

$$G(t,\vec{x}) = \int\frac{d^3k}{(2\pi)^3}\mathrm{e}^{i\vec{k}\cdot\vec{x}}\frac{i}{\tau}\frac{\mathrm{e}^{-i\omega_+t}-\mathrm{e}^{-i\omega_-t}}{\omega_+-\omega_-}\theta(t) = i\theta(t)\mathrm{e}^{-\frac{t}{2\tau}}\int\frac{d^3k}{(2\pi)^3}\mathrm{e}^{i\vec{k}\cdot\vec{x}}\frac{\mathrm{e}^{-i\frac{\sqrt{4Dk^2\tau-1}}{2\tau}t}-\mathrm{e}^{i\frac{\sqrt{4Dk^2\tau-1}}{2\tau}t}}{\sqrt{4Dk^2\tau-1}}. \tag{16}$$

A. One dimensional case

For the preparation of the 3D case, let's consider the 1-D case,

$$G^{1D}(t,x) = i\theta(t)\mathrm{e}^{-\frac{t}{2\tau}}\int\frac{dk}{2\pi}\mathrm{e}^{ikx}\frac{\mathrm{e}^{-i\frac{\sqrt{4Dk^2\tau-1}}{2\tau}t}-\mathrm{e}^{i\frac{\sqrt{4Dk^2\tau-1}}{2\tau}t}}{\sqrt{4Dk^2\tau-1}} \tag{17}$$

Denote $\beta^2\equiv\frac{1}{4D\tau}$ $\frac{D}{\tau}\equiv c^2$, $(\beta=\frac{1}{2c\tau})$,

$$G^{1D}(t,x) = \frac{i}{2\pi}\beta\theta(t)\mathrm{e}^{-\frac{t}{2\tau}}\int dk\mathrm{e}^{ikx}\frac{\mathrm{e}^{-ict\sqrt{k^2-\beta^2}}-\mathrm{e}^{ict\sqrt{k^2-\beta^2}}}{\sqrt{k^2-\beta^2}} = \frac{i}{2\pi}\beta\theta(t)\mathrm{e}^{-\frac{t}{2\tau}}\left[A-B\right], \tag{18}$$

where A and B are defined as

$$A=\int dk\frac{\mathrm{e}^{ikx}}{\sqrt{k^2-\beta^2}}\mathrm{e}^{-ict\sqrt{k^2-\beta^2}},\quad B=\int dk\mathrm{e}^{ikx}\frac{\mathrm{e}^{ikx}}{\sqrt{k^2-\beta^2}}\mathrm{e}^{ict\sqrt{k^2-\beta^2}}. \tag{19}$$

and actually, they can be written as

$$A=\begin{cases}0 & (x>ct)\\ -2i\pi J_0(i\beta\sqrt{c^2t^2-x^2}) & (ct>|x|)\\ -2i\pi J_0(\beta\sqrt{x^2-c^2t^2}) & (-ct>x)\end{cases},\quad B=\begin{cases}0 & (x+ct>0)\\ -2i\pi J_0(\beta\sqrt{x^2-c^2t^2}) & (x+ct<0)\end{cases}, \tag{20}$$

where J_ν is the Bessel function. Then, using $I_0(x)=J_0(ix)$, one can obtain [1, 2]

$$G^{1D}(t,x) = \frac{\mathrm{e}^{-\frac{t}{2\tau}}}{2c\tau}I_0\left(\frac{1}{2c\tau}\sqrt{c^2t^2-x^2}\right)\theta(ct-x)\theta(ct+x). \tag{21}$$

Denote $u\equiv\sqrt{c^2t^2-r^2}$, $J_\nu\equiv J_\nu(i\beta u)$ and $I_\nu\equiv I_\nu(\beta u)$ ($I_\nu(x)$ is the modified Bessel functions of the first kind). The solution for the initial condition $\rho^{1D}(t=0,x)=\delta(x)$ and $\frac{\partial\rho^{1D}}{\partial t}(t=0,x)=0$ can be written as

$$\rho^{1D}(t,x) = \left(1+\tau\frac{\partial}{\partial t}\right)\frac{\mathrm{e}^{-\frac{t}{2\tau}}}{2c\tau}I_0\theta(ct-x)\theta(ct+x) = \frac{\mathrm{e}^{-\frac{t}{2\tau}}}{2c\tau}\left(\frac{1}{2}+\tau\frac{\partial}{\partial t}\right)I_0\theta(ct-x)\theta(ct+x) \tag{22}$$

$$= \frac{\mathrm{e}^{-\frac{t}{2\tau}}}{2c\tau}\left[\frac{1}{2}I_0\theta(ct-x)\theta(ct+x) + c\tau\delta(ct-x)\theta(ct+x)I_0+c\tau\delta(ct+x)\theta(ct-x)I_0+\tau\theta(ct+x)\theta(ct-x)\frac{1}{2c\tau}\left(\frac{c^2t}{u}\right)I_0'\right] \tag{23}$$

$$= \frac{\mathrm{e}^{-\frac{t}{2\tau}}}{2}\left[\frac{1}{2c\tau}I_0\theta(ct-x)\theta(ct+x)+\delta(ct-x)+\delta(ct+x)+\theta(ct+x)\theta(ct-x)\frac{t}{2\tau u}I_1\right], \tag{24}$$

Then,

$$\rho^{1D}(t,x) \;=\; \frac{1}{2}\mathrm{e}^{-\frac{t}{2\tau}}\left\{\delta(ct-x)+\delta(ct+x)+\theta(ct-|x|)\left[\frac{1}{2\tau c}I_0+\frac{t}{2\tau u}I_1\right]\right\}. \tag{25}$$

This is the one which can be found in wikipedia <https://en.wikipedia.org/wiki/Green%27s_function>.

B. Three dimensional case

Denote $r=\sqrt{x^2+y^2+z^2}$, there is a relation between 1-D case and 3-D case when the Green function is isotropic in space,

$$G^{3D}(t,r) \;=\; -\frac{1}{2\pi r}\frac{\partial G^{1D}(t,r)}{\partial r}. \tag{26}$$

Using $\dfrac{\partial u}{\partial r}=\dfrac{-r}{u}$, one can calculate,

$$G^{3D}(t,r) \;=\; -\frac{1}{2c\tau}\frac{\mathrm{e}^{-\frac{t}{2\tau}}}{2\pi r}\frac{\partial}{\partial r}\left[I_0(\beta u)\theta(ct-r)\right]=-\frac{1}{2c\tau}\frac{\mathrm{e}^{-\frac{t}{2\tau}}}{2\pi r}\left[-\delta(ct-r)+\theta(ct-r)\beta\left(\frac{-r}{u}\right)I_0'\right] \tag{27}$$

$$\;=\; \frac{\mathrm{e}^{-\frac{t}{2\tau}}}{4\pi r}\frac{1}{c\tau}\left[\delta(ct-r)+\theta(ct-r)\frac{r}{2c\tau u}I_1\right]. \tag{28}$$

Then, we obtain

$$G^{3D}(t,r) \;=\; \frac{\mathrm{e}^{-\frac{t}{2\tau}}}{4\pi r}\left[\frac{\delta(ct-r)}{c\tau}+\theta(ct-r)\frac{r}{2c^2\tau^2 u}I_1\right]. \tag{29}$$

The solution for the initial condition $\rho^{3D}(t=0,\vec{x})=\delta^{(3)}(\vec{x})$ and $\dfrac{\partial\rho^{3D}}{\partial t}(t=0,\vec{x})=0$ is

$$\rho^{3D}(t,x) \;=\; \left(1+\tau\frac{\partial}{\partial t}\right)G^{3D}(t,r) \tag{30}$$

$$\;=\; \frac{\mathrm{e}^{-\frac{t}{2\tau}}}{4\pi r^2}\left\{\delta(ct-r)\left[1+\frac{t}{2\tau}+\frac{1}{2}\left(\frac{t}{2\tau}\right)^2\right]+\frac{r^2}{(2\tau)^2 c}\theta(ct-r)\left[\frac{1}{cu}I_1+\frac{t}{u^2}I_2\right]\right\}. \tag{31}$$

and the current is

$$\vec{j}^{3D}(t,x) \;=\; -D\vec{e}_r\frac{\partial}{\partial r}G^{3D}(t,r) \tag{32}$$

$$\;=\; \vec{e}_r c\frac{\mathrm{e}^{-\frac{t}{2\tau}}}{4\pi r^2}\left\{\delta(ct-r)\left[1+\frac{t}{2\tau}+\frac{1}{2}\left(\frac{t}{2\tau}\right)^2\right]+\theta(ct-r)\frac{r^3}{c^2(2\tau)^2u^2}I_2\right\} \tag{33}$$

Appendix A: Tips for the modified Bessel function of first kind I_ν

Recurrence Relation

$$I_{\nu+1}(x) = I_{\nu-1}(x) - \frac{2\nu}{x} I_\nu(x). \tag{A1}$$

Derivative:

$$\frac{\partial I_\nu(x)}{\partial x} = \frac{1}{2}(I_{\nu-1}(x) + I_{\nu+1}(x)) = \frac{\nu}{x} I_\nu(x) + I_{\nu+1}(x). \tag{A2}$$

Others:

$$I_{-\nu}(x) = I_\nu(x), \tag{A3}$$

$$I_0(0) = 1,\ I_1(0) = I_2(0) = 0. \tag{A4}$$

Appendix B: Calculation of ρ^{3D}

Using $\dfrac{\partial u}{\partial t} = \dfrac{c^2 t}{u}$, one can calculate

$$
\begin{aligned}
\rho^{3D}(t,x) &= \left(1+\tau\frac{\partial}{\partial t}\right)\frac{\mathrm{e}^{-\frac{t}{2\tau}}}{4\pi r}\left[\frac{\delta(ct-r)}{c\tau}+\theta(ct-r)\frac{r}{2c^2\tau^2 u}I_1\right]. && \text{(B1)}\\
&= \frac{\mathrm{e}^{-\frac{t}{2\tau}}}{4\pi r}\left(\frac{1}{2}+\tau\frac{\partial}{\partial t}\right)\left[\frac{\delta(ct-r)}{c\tau}+\theta(ct-r)\frac{r}{2c^2\tau^2 u}I_1\right] && \text{(B2)}\\
&= \frac{\mathrm{e}^{-\frac{t}{2\tau}}}{4\pi r}\left\{\frac{1}{2}\left[\frac{\delta(ct-r)}{c\tau}+\theta(ct-r)\frac{r}{2c^2\tau^2 u}I_1\right]+\tau\frac{\partial}{\partial t}\left[\frac{\delta(ct-r)}{c\tau}+\theta(ct-r)\frac{r}{2c^2\tau^2 u}I_1\right]\right\} && \text{(B3)}\\
&= \frac{\mathrm{e}^{-\frac{t}{2\tau}}}{4\pi r}\left\{\frac{1}{2}\left[\frac{\delta(ct-r)}{c\tau}+\theta(ct-r)\frac{r}{2c^2\tau^2 u}I_1\right]\right.\\
&\qquad\left.+\delta'(ct-r)+c\tau\delta(ct-r)\frac{r}{2c^2\tau^2 u}I_1+\tau\theta(ct-r)\frac{r}{2c^2\tau^2}\frac{c^2 t}{u}\left[\frac{-1}{u^2}I_1+\frac{1}{u}\beta I_1'\right]\right\} && \text{(B4)}\\
&= \frac{\mathrm{e}^{-\frac{t}{2\tau}}}{4\pi r}\left\{\frac{1}{2}\left[\frac{\delta(ct-r)}{c\tau}+\theta(ct-r)\frac{r}{2c^2\tau^2 u}I_1\right]\right.\\
&\qquad\left.+\delta'(ct-r)+c\tau\delta(ct-r)\frac{r}{2c^2\tau^2 u}I_1+\tau\theta(ct-r)\frac{rt}{2\tau^2 u}\left[\frac{-1}{u^2}I_1+\frac{1}{u}\beta\left(\frac{I_1}{\beta u}+I_2\right)\right]\right\} && \text{(B5)}\\
&= \frac{\mathrm{e}^{-\frac{t}{2\tau}}}{4\pi r}\left\{\frac{1}{2}\left[\frac{\delta(ct-r)}{c\tau}+\theta(ct-r)\frac{r}{2c^2\tau^2 u}I_1\right]\right.\\
&\qquad\left.+\delta'(ct-r)+\delta(ct-r)\frac{r}{2c\tau u}I_1+\theta(ct-r)\frac{rt}{(2c\tau)^2 cu^2}I_2\right\} && \text{(B6)}\\
&= \frac{\mathrm{e}^{-\frac{t}{2\tau}}}{4\pi r^2}\left\{r\delta'(ct-r)+\delta(ct-r)\left[\frac{r}{2\tau c}+\frac{r^2}{2c\tau u}I_1\right]+r^2\theta(ct-r)\left[\frac{1}{(2\tau)^2c^2u}I_1+\frac{t}{(2\tau)^2cu^2}I_2\right]\right\} && \text{(B7)}\\
&= \frac{\mathrm{e}^{-\frac{t}{2\tau}}}{4\pi r^2}\left\{r\delta'(ct-r)+\delta(ct-r)\left[\frac{ct}{2\tau c}+\frac{c^2t^2}{2c\tau}\beta\frac{I_1}{\beta u}\right]+\frac{r^2}{(2\tau)^2c}\theta(ct-r)\left[\frac{1}{cu}I_1+\frac{t}{u^2}I_2\right]\right\} && \text{(B8)}\\
&= \frac{\mathrm{e}^{-\frac{t}{2\tau}}}{4\pi r^2}\left\{r\delta'(ct-r)+\delta(ct-r)\left[\frac{ct}{2\tau c}+\frac{c^2t^2}{2c\tau}\beta\frac{1}{2}\right]+\frac{r^2}{(2\tau)^2c}\theta(ct-r)\left[\frac{1}{cu}I_1+\frac{t}{u^2}I_2\right]\right\} && \text{(B9)}
\end{aligned}
$$

[Here, the L'Hôpital's rule is applied for $\lim_{z\to 0}\dfrac{I_1(z)}{z} = \lim_{z\to 0}\dfrac{dI_1(z)}{dz} = \dfrac{1}{2}(I_0(0)+I_2(0))$]

$$
\begin{aligned}
&= \frac{\mathrm{e}^{-\frac{t}{2\tau}}}{4\pi r^2}\left\{r\delta'(ct-r)+\delta(ct-r)\left[\frac{t}{2\tau}+\frac{1}{2}\left(\frac{t}{2\tau}\right)^2\right]+\frac{r^2}{(2\tau)^2c}\theta(ct-r)\left[\frac{1}{cu}I_1+\frac{t}{u^2}I_2\right]\right\} && \text{(B10)}\\
&= \frac{\mathrm{e}^{-\frac{t}{2\tau}}}{4\pi r^2}\left\{\delta(ct-r)\left[1+\frac{t}{2\tau}+\frac{1}{2}\left(\frac{t}{2\tau}\right)^2\right]+\frac{r^2}{(2\tau)^2c}\theta(ct-r)\left[\frac{1}{cu}I_1+\frac{t}{u^2}I_2\right]\right\} && \text{(B11)}
\end{aligned}
$$

[Assumed the integration with r^2dr for the transfromation for the derivative of the delta function]

Here, the integration of n with r^2dr gives the flux in t-direction, which is the quantity we are supposed to estimate in the simulation.

Appendix C: Calculation of $\vec{j}^{3D}$

$$\frac{\partial}{\partial r}\frac{\mathrm{e}^{-\frac{t}{2\tau}}}{4\pi r}\left[\frac{\delta(ct-r)}{c\tau}+\theta(ct-r)\frac{r}{2c^2\tau^2 u}I_1\right] \tag{C1}$$

$$= \frac{\mathrm{e}^{-\frac{t}{2\tau}}}{4\pi r^2}\left(-1+r\frac{\partial}{\partial r}\right)\left[\frac{\delta(ct-r)}{c\tau}+\theta(ct-r)\frac{r}{2c^2\tau^2 u}I_1\right] \tag{C2}$$

$$= \frac{\mathrm{e}^{-\frac{t}{2\tau}}}{4\pi r^2}\left\{-\frac{\delta(ct-r)}{c\tau}-\theta(ct-r)\frac{r}{2c^2\tau^2 u}I_1-r\frac{\delta'(ct-r)}{c\tau}\right.$$
$$\left.-\delta(ct-r)\frac{r^2}{2c^2\tau^2 u}I_1+\theta(ct-r)\frac{r^2}{2c^2\tau^2}\left[\frac{I_1}{ru}+\left(\frac{-r}{u}\right)\left(\frac{-1}{u^2}I_1+\frac{1}{u}\beta I_1'\right)\right]\right\} \tag{C3}$$

$$= \frac{\mathrm{e}^{-\frac{t}{2\tau}}}{4\pi r^2}\left\{-\frac{\delta(ct-r)}{c\tau}-r\frac{\delta'(ct-r)}{c\tau}-\delta(ct-r)\frac{r^2}{2c^2\tau^2 u}I_1+\theta(ct-r)\frac{r^3}{2c^2\tau^2u^2}\left[\frac{1}{u}I_1-\beta\left(\frac{I_1}{\beta u}+I_2\right)\right]\right\} \tag{C4}$$

$$= \frac{\mathrm{e}^{-\frac{t}{2\tau}}}{4\pi r^2}\left[-\frac{\delta(ct-r)}{c\tau}-r\frac{\delta'(ct-r)}{c\tau}-\delta(ct-r)\frac{r^2}{2c^2\tau^2 u}I_1-\theta(ct-r)\frac{\beta r^3}{2c^2\tau^2u^2}I_2\right] \tag{C5}$$

$$= \frac{\mathrm{e}^{-\frac{t}{2\tau}}}{4\pi r^2}\left\{r\frac{\delta'(ct-r)}{c\tau}+\delta(ct-r)\left[\frac{1}{c\tau}+\frac{r^2}{2c^2\tau^2}\beta\frac{I_1}{\beta u}\right]+\theta(ct-r)\frac{\beta r^3}{2c^2\tau^2u^2}I_2\right\} \tag{C6}$$

$$= \frac{\mathrm{e}^{-\frac{t}{2\tau}}}{4\pi r^2}\left\{r\frac{\delta'(ct-r)}{c\tau}+\delta(ct-r)\left[\frac{1}{c\tau}+\frac{r^2}{2c^2\tau^2}\frac{\beta}{2}\right]+\theta(ct-r)\frac{\beta r^3}{2c^2\tau^2u^2}I_2\right\} \tag{C7}$$

$$= \frac{\mathrm{e}^{-\frac{t}{2\tau}}}{4\pi r^2}\left\{\delta(ct-r)\left[\frac{1}{c\tau}\left(-\frac{r}{c}\right)\left(-\frac{1}{2\tau}\right)+\frac{1}{c\tau}+\frac{1}{2c\tau}\left(\frac{t}{2\tau}\right)^2\right]+\theta(ct-r)\frac{r^3}{4c^3\tau^3u^2}I_2\right\} \tag{C8}$$

$$= \frac{1}{c\tau}\frac{\mathrm{e}^{-\frac{t}{2\tau}}}{4\pi r^2}\left\{\delta(ct-r)\left[1+\frac{t}{2\tau}+\frac{1}{2}\left(\frac{t}{2\tau}\right)^2\right]+\theta(ct-r)\frac{r^3}{4c^2\tau^2u^2}I_2\right\} \tag{C9}$$

[Assumed the integration with dt for the transfromation for the derivative of the delta function]

Here, the integration of $\vec{j}$ with time dt gives the flux in the radial direction, which is the quantity we are supposed to estimate in the simulation.

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[1] P. Romatschke, Int. J. Mod. Phys. E **19**, 1 (2010) doi:10.1142/S0218301310014613 [arXiv:0902.3663 [hep-ph]].
[2] P. M. Morse and H. Feshbach "Methods of theoretical physics," part I, McGraw-Hill (1963).